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Buckling Theory in Solid Structure with Small Thickness : Part 2

Application to the Externally Pressurized Cylindrical Shell Buckling

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A consistent methodology for the analysis of buckling phenomena in three dimensional solids developed in Part 1 is applied to a simple structure, i.e., the externally pressurized cylindrical shell structure. The primary state of the shell is investigated analytically, using asymptotic technique, and then the straightforward buckling analysis is followed up to the third order, adopting the general stability theory in Part 1. The full analysis is done through the analytical manner. Hence, the closed form solution is obtained. Finally, the result is compared with the classical one.

Key Words : Strain Energy Density Function, Asymptotic Physical Component, Hypoelastic Material, Incremental Moduli, Load Parameter, Characteristic Equation, Mode Orthogonality, Wave Number

1. Introduction

Succeeding the previous work (i.e., the general stability theory development in Part 1 (Kwon, 1992)), the controversial structural stability problem, i. e., the externally pressurized cylindrical shell buckling problem, is treated, adopting the general stability theory developed in Part 1 (Kwon, 1992). The same problem was solved by Timosenko(1961), using the classical nonlinear shell theory with limitations. Similar other problems were treated by Nash(1955), Donnell(1956), Galletly and Bart(1956), Singer(1960), Sobel (1964) and Cheng et al. (1971). Thereafter, Triantafyllidis and Kwon(1987) solved the same problem for the incompressible material shell structure, also using the asymptotic technique. Here, the compressible isotropic hypoelastic cylindrical shell is pressurized externally, and then the forthcoming buckling phenomenon is analysed.

The prebuckling state is solved in an asymptotic manner in the sense that the full analytical primary solution is not required for the asymptotic instabiltity analysis. The stability analysis is done up to the third order. Hence, the four term critical load parameter expansion is obtained, and the corresponding five term critical load (external hydrostatic pressure) expansion is obtained with respect to the thickness parameter.

The imperfection effect on the buckling is not considered in this analysis.

2. The Primary State

The primary state for the pressurized long cylindrical shell is just a plane strain one if the shell is long enough. Both ends of the cylindrical shell are assumed to be stress free, but a proper forced boundary condition is applied to prevent the rigid body motion. And the principal axes of strain remain fixed with respect to the material point. Hence, the principal stretch ratios of a material point, whose distances from the cylinder axis in the initial (stress free) and in the current configuration are denoted by R and r respectively, are given by

$$\lambda_\theta=\frac{r}{R},\ \lambda_z=1,\ \lambda_r=\frac{dr}{dR}, \tag{1}$$

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in the cylindrical coordinate (r, θ, z), Now, we have to choose a proper coordinate system (θ^α, ζ) for the application of the general theory developed in Part 1. Here, the natural selection of (θ^α, ζ) may be

$$\theta^1=\theta,\quad \theta^2=z,\quad \theta^3=\zeta=r-(r_i+h/2), \tag{2}$$

where r_i is the inner radius of the deformed current cylindrical shell and h is the deformed current shell thickness. Hence, we take the deformed current configuration as the reference one. Therefore, the principal stretch ratios in (1) can be written in terms of (θ^α, ζ) as

$$\lambda_1=(\zeta+r_i+h/2)/R,\quad \lambda_2=1,\quad \lambda_3=d\zeta/dR. \tag{3}$$

Now, for the compressible isotropic material, the constitutive equation is

$$\tau_i=J\sigma_i=\lambda_i\frac{\partial W}{\partial\lambda_i}\ \text{(no sum)}\ (J=\lambda_1\lambda_2\lambda_3), \tag{4}$$

where $W=W(\lambda_1, \lambda_2, \lambda_3)$ is a strain energy density function, σ_i is the principal Cauchy stress in the deformed current configuration and τ_i is the principal Kirchhoff stress. Here, we consider only a simple material whose uniaxial stress-strain behavior is a piecewise power-law. A strain energy density function W which describes such a material is

$$W(\lambda_1, \lambda_2, \lambda_3)=E\varepsilon_y^2\left[\frac{\chi}{\chi+1}\left(\frac{\tau_e}{\tau_y}\right)^{\chi+1}-\frac{1-2\nu}{6}\left(\frac{\tau_e}{\tau_y}\right)^2\right]+\frac{E}{6(1-2\nu)}(\varepsilon_1+\varepsilon_2+\varepsilon_3)^2+C, \tag{5}$$

in which the equivalent Kirchhoff stress τ_e is related to the equivalent logarithmic strain ε_e by the relation

$$\left(\frac{\varepsilon_e}{\varepsilon_y}\right)=\left(\frac{\tau_e}{\tau_y}\right)^\chi-\frac{1-2\nu}{3}\left(\frac{\tau_e}{\tau_y}\right), \tag{6}$$

where $\chi=1$ if $\varepsilon_e\leq 2(1+\nu)\varepsilon_y/3$, $\chi=m$ if $\varepsilon_e>2(1+\nu)\varepsilon_y/3$, $\varepsilon_y=\tau_y/E$ is the initial yield strain, τ_y is the initial yield stress, and m is the hardening exponent. Also, E is the Young's modulus, ν is the Poisson's ratio and the constant C which depends on E, ν, m, and ε_y is constructed so that it assures the continuity of W at $\varepsilon_e=2(1+\nu)\varepsilon_y/3$. The equivalent logarithmic strain ε_e is expressed in terms of the principal logarithmic strains $\varepsilon_i=\ln\lambda_i$ by the relation

$$\varepsilon_e=\frac{2}{3}(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2-\varepsilon_1\varepsilon_2-\varepsilon_2\varepsilon_3-\varepsilon_3\varepsilon_1)^{\frac{1}{2}}. \tag{7}$$

For the isochoric deformations, the limit $\nu\to 0.5$ of the above expression for the strain energy density function yields the incompressible description. Now using (5), (4) may take the following form

$$\tau_i=E\varepsilon_y\left(\frac{\tau_e}{\tau_y}\right)\frac{\partial\varepsilon_e}{\partial\varepsilon_i}+\frac{E}{3(1-2\nu)}(\varepsilon_1+\varepsilon_2+\varepsilon_3). \tag{8}$$

We treat only the case of $m=1$ in this study, since the buckling phenomenon of the externally pressurized long cylindrical shell is assumed to occur in the elastic range. Now, the only nontrivial equilibrium equation in the cylindrical coordinates is, using the physical components,

$$\frac{\partial\sigma_3}{\partial r}+\frac{\sigma_3-\sigma_1}{r}=0$$
$$\text{with}\quad \sigma_3=0\ \text{at}\ r=r_i,\ \sigma_3=-p_o\ \text{at}\ r=r_o, \tag{9}$$

for the externally pressurized case. The Eq. (9) is usually the highly non-linear differential equation whose solution is extremely difficult to obtain analytically. However, we can solve the above Eq. (9) asymptotically in the sense that we don't need the full analytical primary solution for the asymptotic bifurcation analysis. We write the Eq. (9) in terms of the coordinate system (θ^α, ζ) as

$$\frac{\partial\sigma_3}{\partial\xi}+\varepsilon\left\{\left(\xi+\frac{1}{2}\right)\frac{\partial\sigma_3}{\partial\xi}+\sigma_3-\sigma_1\right\}=0$$
$$\text{with}\quad \xi=\zeta/\varepsilon r_i,\ \varepsilon\equiv h/r_i, \tag{10}$$

where $h=r_o-r_i$ is the constant current shell thickness and we take r_i as the reference length(L) of the current cylinder shell configuration. And so $\varepsilon=h/r_i$ is the current thickness parameter. And also, we have the corresponding boundary condition as

$$\sigma_3\left(-\frac{1}{2}\right)=0,\ \sigma_3\left(\frac{1}{2}\right)=-p_o$$
$$\text{with}\quad \sigma_3=\sigma_3(\xi\,;\,\lambda(\varepsilon),\,\varepsilon)\quad -\frac{1}{2}\leq\xi\leq\frac{1}{2}, \tag{11}$$

where $\lambda\equiv r_i/R_i$ is defined as the load parameter and R_i is the initial inner radius of the cylindrical shell. For the asymptotic solution, we may have the following expansion

$$R^*(\xi,\,\varepsilon)\equiv\frac{R}{R_i}=\overset{0}{R}(\xi)+\overset{1}{R}(\xi)\,\varepsilon+\overset{2}{R}(\xi)\,\varepsilon^2+\overset{3}{R}(\xi)\,\varepsilon^3+\dots$$

with $R^*(-\frac{1}{2};\,\varepsilon)=1$

$$\lambda(\varepsilon)=\overset{0}{\lambda}+\overset{1}{\lambda}\varepsilon+\overset{2}{\lambda}\varepsilon^2+\overset{3}{\lambda}\varepsilon^3+\dots \qquad (12)$$

and the corresponding stress expansions are assumed to be

$$\sigma_1=\overset{0}{\sigma}_1+\overset{1}{\sigma}_1\varepsilon+\overset{2}{\sigma}_1\varepsilon^2+\dots$$
$$\sigma_2=\overset{0}{\sigma}_2+\overset{1}{\sigma}_2\varepsilon+\overset{2}{\sigma}_2\varepsilon^2+\dots$$
$$\sigma_3=\overset{0}{\sigma}_3+\overset{1}{\sigma}_3\varepsilon+\overset{2}{\sigma}_3\varepsilon^2+\dots$$
$$p_o=\overset{0}{p}_o+\overset{1}{p}_o\varepsilon+\overset{2}{p}_o\varepsilon^2+\dots$$

Hence, we are ready to obtain the asymptotic prebuckling solution. The general higher order solutions are quite complicate, and so we just record up to the first order solution.

The lowest order solution is obtained as follows.

The ε^0 term of the Eq. (10) gives us the following solution, using the corresponding boundary condition

$$\overset{0}{\sigma}_1=\frac{E}{(1-\nu^2)\,\overset{0}{\lambda}{}^{\frac{1-2\nu}{1-\nu}}\ln\overset{0}{\lambda}},$$
$$\overset{0}{\sigma}_2=\frac{E\nu}{(1-\nu^2)\,\overset{0}{\lambda}{}^{\frac{1-2\nu}{1-\nu}}\ln\overset{0}{\lambda}},$$
$$\overset{0}{\sigma}_3=0, \qquad (13)$$

with $\overset{0}{p}_o=0$

And, correspondingly, we may get

$$\overset{0}{R}=1,\ \frac{\partial\overset{1}{R}}{\partial\xi}=\overset{0}{\lambda}{}^{\frac{1}{1-\nu}}\ \text{or}\ \overset{1}{R}=\overset{0}{\lambda}{}^{\frac{1}{1-\nu}}(\xi+\frac{1}{2}).$$

The first order solution is obtained as follows.

From the ε^1 term of the Eq. (10), we may obtain the following solution, using the corresponding boundary condition,

$$\overset{1}{\sigma}_1=\frac{E}{(1-\nu^2)\,\overset{0}{\lambda}{}^{\frac{1-2\nu}{1-\nu}}}[\,(1-\frac{1-2\nu}{1-\nu}\ln\overset{0}{\lambda})\{\frac{\overset{1}{\lambda}}{\overset{0}{\lambda}}+(1-\overset{0}{\lambda}{}^{\frac{1}{1-\nu}})(\xi+\frac{1}{2})\}+\{\frac{\nu}{1-\nu}-\frac{1-2\nu}{(1-\nu)^2}\ln\overset{0}{\lambda}\}(\ln\overset{0}{\lambda})(\xi+\frac{1}{2})\,],$$

$$\overset{1}{\sigma}_2=\frac{E\nu}{(1-\nu^2)\,\overset{0}{\lambda}{}^{\frac{1-2\nu}{1-\nu}}}(1-\frac{1-2\nu}{1-\nu}\ln\overset{0}{\lambda})\{\frac{\overset{1}{\lambda}}{\overset{0}{\lambda}}+(1-\overset{0}{\lambda}{}^{\frac{1}{1-\nu}}+\frac{1}{1-\nu}\ln\overset{0}{\lambda})(\xi+\frac{1}{2})\},$$

$$\overset{1}{\sigma}_3=\frac{E}{(1-\nu^2)\,\overset{0}{\lambda}{}^{\frac{1-2\nu}{1-\nu}}}(\ln\overset{0}{\lambda})(\xi+\frac{1}{2})\},$$

with,

$$\overset{1}{p}_o=-\overset{1}{\sigma}_3(\frac{1}{2})=-\frac{E}{(1-\nu^2)\,\overset{0}{\lambda}{}^{\frac{1-2\nu}{1-\nu}}}\ln\overset{0}{\lambda}. \qquad (14)$$

And also, we may get

$$\overset{2}{R}=\overset{0}{\lambda}{}^{\frac{1}{1-\nu}}(\xi+\frac{1}{2})[\frac{1}{1-\nu}\frac{\overset{1}{\lambda}}{\overset{0}{\lambda}}+\frac{1}{2}\{\frac{\nu}{1-\nu}(1-\overset{0}{\lambda}{}^{\frac{1}{1-\nu}})-\frac{1-2\nu}{(1-\nu)^2}\ln\overset{0}{\lambda}\}(\xi+\frac{1}{2})].$$

Similarly, we may obtain the higher order solutions. For example, from the ε^2 term of the Eq. (10), we can obtain the following solution, at the early stage,

$$\overset{2}{\sigma}_3=\frac{E}{(1-\nu^2)\,\overset{0}{\lambda}{}^{\frac{1-2\nu}{1-\nu}}}(\xi+\frac{1}{2})[\,(1-\frac{1-2\nu}{1-\nu}\ln\overset{0}{\lambda})\{\frac{\overset{1}{\lambda}}{\overset{0}{\lambda}}+\frac{1}{2}(1-\overset{0}{\lambda}{}^{\frac{1}{1-\nu}})(\xi+\frac{1}{2})\}+\frac{1}{2}\{\frac{-2+3\nu}{1-\nu}-\frac{1-2\nu}{(1-\nu)^2}\ln\overset{0}{\lambda}\}(\ln\overset{0}{\lambda})(\xi+\frac{1}{2})\,], \qquad (15)$$

and hence,

$$\overset{2}{p}_o=-\overset{2}{\sigma}_3(\frac{1}{2})$$
$$=-\frac{E}{(1-\nu^2)\,\overset{0}{\lambda}{}^{\frac{1-2\nu}{1-\nu}}}[\,(1-\frac{1-2\nu}{1-\nu}\ln\overset{0}{\lambda})\{\frac{\overset{1}{\lambda}}{\overset{0}{\lambda}}+\frac{1}{2}(1-\overset{0}{\lambda}{}^{\frac{1}{1-\nu}})\}+\frac{1}{2}\{\frac{-2+3\nu}{1-\nu}-\frac{1-2\nu}{(1-\nu)^2}\ln\overset{0}{\lambda}\}\ln\overset{0}{\lambda}\,]$$

etc.

3. The Constitutive Law

For the STÖREN-RICE hypoelastic material, the incremental moduli L^{ijkl} take, especially for $m=1$,

$$\mathbf{L}^{ijkl}=\frac{E}{(1+\nu)J}\{\frac{1}{2}(g^{ik}g^{jl}+g^{jk}g^{il})+\frac{\nu}{1-2\nu}g^{ij}g^{kl}\}-\frac{1}{2}(g^{ik}\sigma^{jl}+g^{jl}\sigma^{ik}+g^{ik}\sigma^{il}+g^{il}\sigma^{jk}). \quad (16)$$

Or, using the physical components,

$\sigma^{11}=\sigma_1/r^2$, $\sigma^{22}=\sigma_2$, $\sigma^{33}=\sigma_3$ and $\sigma^{ij}=0$ for $i\neq j$, we may have

$$\begin{aligned}
\mathbf{L}^{1111}&=\{\frac{EI(1-\nu)}{(1+\nu)(1-2\nu)}-2\sigma_1\}\frac{1}{r^4},\\
\mathbf{L}^{2222}&=\frac{EI(1-\nu)}{(1+\nu)(1-2\nu)}-2\sigma_2,\\
\mathbf{L}^{3333}&=\frac{EI(1-\nu)}{(1+\nu)(1-2\nu)}-2\sigma_3,\\
\mathbf{L}^{1122}&=\mathbf{L}^{1133}\frac{EI\nu}{(1+\nu)(1-2\nu)}\frac{1}{r^2},\\
\mathbf{L}^{2233}&=\frac{EI\nu}{(1+\nu)(1-2\nu)},\\
\mathbf{L}^{1212}&=\frac{1}{2}\{\frac{EI}{1+\nu}-(\sigma_1+\sigma_2)\}\frac{1}{r^2},\\
\mathbf{L}^{1313}&=\frac{1}{2}\{\frac{EI}{1+\nu}-(\sigma_3+\sigma_1)\}\frac{1}{r^2},\\
\mathbf{L}^{2323}&=\frac{1}{2}\{\frac{EI}{1+\nu}-(\sigma_2+\sigma_3)\}, \qquad (17)
\end{aligned}$$

and all others are zero, where we defined $I=1/J=1/\lambda_1\lambda_2\lambda_3$ and we used the following current metric tensor

$$g^{11}=\frac{1}{r^2},\ g^{22}=g^{33}=1 \text{ and } g^{ij}=0 \text{ for } i\neq j.$$

Now the corresponding prebuckling moduli are, using the physical components of the Cauchy stress,

$$\begin{aligned}
L^{1111}&=\{\frac{EI(1-\nu)}{(1+\nu)(1-2\nu)}-\sigma_1\}\frac{1}{r^4},\\
L^{2222}&=\frac{EI(1-\nu)}{(1+\nu)(1-2\nu)}-\sigma_2,\\
L^{3333}&=\frac{EI(1-\nu)}{(1+\nu)(1-2\nu)}-\sigma_3,\\
L^{1122}&=L^{1133}=\frac{EI\nu}{(1+\nu)(1-2\nu)}\frac{1}{r^2},\\
L^{2233}&=\frac{EI\nu}{(1+\nu)(1-2\nu)},\\
L^{1212}&=\frac{1}{2}\{\frac{EI}{1+\nu}+(\sigma_1-\sigma_2)\}\frac{1}{r^2},\\
L^{1313}&=\frac{1}{2}\{\frac{EI}{1+\nu}+(\sigma_1-\sigma_3)\}\frac{1}{r^2},\\
L^{2121}&=\frac{1}{2}\{\frac{EI}{1+\nu}+(\sigma_2-\sigma_1)\}\frac{1}{r^2},\\
L^{2323}&=\frac{1}{2}\{\frac{EI}{1+\nu}+(\sigma_2-\sigma_3)\},\\
L^{3131}&=\frac{1}{2}\{\frac{EI}{1+\nu}+(\sigma_3-\sigma_1)\}\frac{1}{r^2},\\
L^{3232}&=\frac{1}{2}\{\frac{EI}{1+\nu}+(\sigma_3-\sigma_2)\},\\
L^{2112}&=\frac{1}{2}\{\frac{EI}{1+\nu}-(\sigma_1+\sigma_2)\}\frac{1}{r^2}\},\\
L^{3113}&=\frac{1}{2}\{\frac{EI}{1+\nu}-(\sigma_3+\sigma_1)\}\frac{1}{r^2},\\
L^{3223}&=\frac{1}{2}\{\frac{EI}{1+\nu}-(\sigma_2+\sigma_3)\}, \qquad (18)
\end{aligned}$$

and all others are zero.

Also we have, noting that $p_o=-\sigma_3(r_o)$,

$$\begin{aligned}
N^{1122}&=N^{2211}=N^{1133}=N^{3311}=-\frac{\sigma_3}{r^2},\\
N^{1221}&=N^{2112}=N^{1331}=N^{3113}=\frac{\sigma_3}{r^2},\\
N^{2233}&=N^{3322}=-\sigma_3,\ N^{2332}=N^{3223}=\sigma_3, \qquad (19)
\end{aligned}$$

and all others are zero. These moduli are defined as,

$$\begin{aligned}
L^{ijkl}&=\mathbf{L}^{ijkl}+\sigma^{ik}g^{jl}=L^{klij}\\
N^{ijkl}&=p_o(g^{ij}g^{kl}-g^{iL}g^{jk}=N^{klij},
\end{aligned}$$

and assmmed as,

$$\begin{aligned}
\mathbf{L}^{ijkl}&=\overset{0}{\mathbf{L}}{}^{ijkl}+\overset{1}{\mathbf{L}}{}^{ijkl}\varepsilon+\overset{2}{\mathbf{L}}{}^{ijkl}\varepsilon^2+\overset{3}{\mathbf{L}}{}^{ijkl}\varepsilon^3+\dots\\
L^{ijkl}&=\overset{0}{L}{}^{ijkl}+\overset{1}{L}{}^{ijkl}\varepsilon+\overset{2}{L}{}^{ijkl}\varepsilon^2+\overset{3}{L}{}^{ijkl}\varepsilon^3+\dots\\
N^{ijkl}&=\overset{0}{N}{}^{ijkl}+\overset{1}{N}{}^{ijkl}\varepsilon+\overset{2}{N}{}^{ijkl}\varepsilon^2+\overset{3}{N}{}^{ijkl}\varepsilon^3+\dots
\end{aligned}$$

We should note the following relations for the above assumptions in this particular case,

$$\begin{aligned}
r&=r_i\{1+\varepsilon(\xi+\frac{1}{2})\},\\
\frac{\partial r}{\partial\varepsilon}&=r_i(\xi+\frac{1}{2}),\ \frac{\partial^2 r}{\partial\varepsilon^2}=\dots=0,\\
\frac{1}{r^2}&=\frac{1}{r_i^2}\{1-2(\xi+\frac{1}{2})\varepsilon+3(\xi+\frac{1}{2})^2\varepsilon^2-4(\xi+\frac{1}{2})^3\varepsilon^3+5(\xi+\frac{1}{2})^4\varepsilon^4+\dots\},\\
\frac{1}{r^4}&=\frac{1}{r_i^4}\{1-4(\xi+\frac{1}{2})\varepsilon+10(\xi+\frac{1}{2})^2\varepsilon^2-20(\xi+\frac{1}{2})^3\varepsilon^3+\dots\}, \qquad (20)
\end{aligned}$$

and with the following expansions of the Christoffel symbols,

$$\Gamma^i_{jk}=\overset{0}{\Gamma}{}^i_{jk}+\overset{1}{\Gamma}{}^i_{jk}\varepsilon+\overset{2}{\Gamma}{}^i_{jk}\varepsilon^2+\overset{3}{\Gamma}{}^i_{jk}\varepsilon^3+\dots$$

with $\Gamma^1_{31}=\Gamma^1_{13}=\frac{1}{r}$, $\Gamma^3_{11}=-r$, and others are zero.

Or, we have

$$\overset{0}{\Gamma}{}^1_{31}=\frac{1}{r_i},\ \overset{1}{\Gamma}{}^1_{31}=-\frac{1}{r_i}(\xi+\frac{1}{2}),\ \overset{2}{\Gamma}{}^1_{31}=\frac{1}{r_i}(\xi+\frac{1}{2})^2,$$
$$\overset{3}{\Gamma}{}^1_{31}=-\frac{1}{r_i}(\xi+\frac{1}{2})^3,$$

etc.,

$$\overset{0}{\Gamma}{}^3_{11}=-r_i,\ \overset{1}{\Gamma}{}^3_{11}=-r_i(\xi+\frac{1}{2}),\ \overset{2}{\Gamma}{}^3_{11}=\overset{3}{\Gamma}{}^3_{11}=\dots=0,$$

and others are zero.

4. The Stability Analysis

In this section, we obtain the asymptotic stability solution for the pressurized long cylindrical shell, applying the general stability theory developed in Part 1.

The lowest order analysis is as follows.

Since the current cylindrical shell's thickness h is constant, the lowest order general solution is,

$$(\overset{0}{G}{}^{\alpha i\beta l}\overset{0}{u}_{l,\beta})\overset{0}{|}_{\alpha}-\overset{0}{F}{}_1^{3i\beta l}\overset{0}{u}_{l,\beta}=0,$$

with
$$\overset{0}{G}{}^{\alpha j\beta l}=\overset{0}{L}{}^{\alpha j\beta l}-\overset{0}{L}{}^{\alpha j3k}(\overset{0}{L}{}^{3m3k})^{-1}\overset{0}{L}{}^{3m\beta l}$$
$$\overset{0}{F}{}^{ij\beta l}=\overset{1}{N}{}^{ij\beta l}-\overset{1}{N}{}^{ij3k}(\overset{0}{L}{}^{3r3k})^{-1}\overset{0}{L}{}^{3r\beta l}$$
$$\equiv\overset{0}{F}{}_1^{ij\beta l}\xi+\overset{0}{F}{}_2^{ij\beta l}. \tag{21}$$

Now, we define the asymptotic physical components as, for the convenience,

$$\overset{0}{u}_1=r_i\overset{0}{v}_\theta,\quad \overset{0}{u}_2=\overset{0}{v}_z,\quad \overset{0}{u}_3=\overset{0}{v}_r. \tag{22}$$

Using these, we may have, and

$$\overset{0}{T}{}^{\alpha j}\overset{0}{|}_{\alpha}=r_i(\frac{\partial\overset{0}{T}{}^{\alpha j}}{\partial\theta^\alpha}+\frac{1}{r_i}\overset{0}{T}{}^{3j}-r_i\delta^j_3\overset{0}{T}{}^{11}+\frac{1}{r_i}\delta^j_1\overset{0}{T}{}^{13}).$$

And the lowest order moduli are

$$\overset{0}{G}{}^{1111}=\overset{0}{\Delta}(1+\alpha)(1-\ln\overset{0}{\lambda})r_i^4,$$
$$\overset{0}{G}{}^{1122}=\overset{0}{\Delta}\alpha r_i^2,$$
$$\overset{0}{G}{}^{2112}=\overset{0}{G}{}^{1221}=\frac{1}{2}\overset{0}{\Delta}\{1-(1+2\alpha)\ln\overset{0}{\lambda}\}r_i^2,$$
$$\overset{0}{G}{}^{2121}=\frac{1}{2}\overset{0}{\Delta}(1-\ln\overset{0}{\lambda})r_i^2,$$
$$\overset{0}{G}{}^{1212}=\frac{1}{2}\overset{0}{\Delta}(1+\ln\overset{0}{\lambda})r_i^2,$$
$$\overset{0}{G}{}^{2222}=\overset{0}{\Delta}(1+\alpha)(1-\nu\ln\overset{0}{\lambda}),$$
$$\overset{0}{G}{}^{1313}=\overset{0}{\Delta}(1+\alpha)(\ln\overset{0}{\lambda})r_i^2,$$
$$\overset{0}{G}{}^{2323}=\overset{0}{\Delta}\alpha\ln\overset{0}{\lambda}, \tag{23}$$

and also we have

$$\overset{0}{F}{}_1^{3113}=\overset{0}{F}{}_1^{1331}=\overset{0}{\Delta}(1+\alpha)(\ln\overset{0}{\lambda})r_i^2,$$
$$\overset{0}{F}{}_1^{3223}=\overset{0}{F}{}_1^{2332}=\overset{0}{\Delta}(1+\alpha)\ln\overset{0}{\lambda},$$
$$\overset{0}{F}{}_1^{1133}=\overset{0}{F}{}_1^{3311}=-\overset{0}{\Delta}(1+\alpha)(\ln\overset{0}{\lambda})r_i^2,$$
$$\overset{0}{F}{}_1^{2233}=\overset{0}{F}{}_1^{3322}=-\overset{0}{\Delta}(1+\alpha)\ln\overset{0}{\lambda},$$

with $\overset{0}{\Delta}\equiv\frac{E}{(1+\nu)\overset{0}{\lambda}{}^{1-\alpha}}$, $\alpha\equiv\frac{\nu}{1-\nu}$.

Then, the Eq. (21) may take the following form, with the change of variable as $x\equiv z/r_i$

$$(1+\alpha)(1-\ln\overset{0}{\lambda})\frac{\partial^2\overset{0}{v}_\theta}{\partial\theta^2}+\frac{1}{2}(1-\ln\overset{0}{\lambda})\frac{\partial^2\overset{0}{v}_\theta}{\partial x^2}+\frac{1+2\alpha}{2}(1-\ln\overset{0}{\lambda})\frac{\partial^2\overset{0}{v}_z}{\partial x\partial\theta}+(1+\alpha)(1-\ln\overset{0}{\lambda})\frac{\partial\overset{0}{v}_r}{\partial\theta}=0,$$

$$\frac{1+2\alpha}{2}(1-\ln\overset{0}{\lambda})\frac{\partial^2\overset{0}{v}_\theta}{\partial x\partial\theta}+\frac{1}{2}(1+\ln\overset{0}{\lambda})\frac{\partial^2\overset{0}{v}_z}{\partial\theta^2}+(1+\alpha)(1-\nu\ln\overset{0}{\lambda})\frac{\partial^2\overset{0}{v}_z}{\partial x^2}+\{\alpha-(1+\alpha)\ln\overset{0}{\lambda}\}\frac{\partial\overset{0}{v}_r}{\partial x}=0,$$

$$(1+\alpha)(1-\ln\overset{0}{\lambda})\frac{\partial\overset{0}{v}_\theta}{\partial\theta}+\{\alpha-(1+\alpha)1\overset{0}{\lambda}\}\frac{\partial\overset{0}{v}_z}{\partial x}-(1+\alpha)(\ln\overset{0}{\lambda})\frac{\partial^2\overset{0}{v}_r}{\partial\theta^2}-\alpha(\ln\overset{0}{\lambda})\frac{\partial^2\overset{0}{v}_r}{\partial x^2}+(1+\alpha)(1-2\ln\overset{0}{\lambda})\overset{0}{v}_r=0, \tag{24}$$

which is the desired coupled partial differential equation governing the lowest order mode. In order to solve the above equations, we should note that the mode is periodic in the angualar direction for the long cylindrical shell. And hence, the solution should be the following form for the nonaxisymmetric buckling.

$$\overset{0}{v}_\theta(\theta,x)=\overset{0}{u}_n(x)\sin n\theta,$$
$$\overset{0}{v}_z(\theta,x)=\overset{0}{v}_n(x)\cos n\theta,$$
$$\overset{0}{v}_r(\theta,x)=\overset{0}{w}_n(x)\cos n\theta, \tag{25}$$

where $n(\neq0)$ is the angular wavenumber. Inserting (25) into (24), we get the following equation,

$$(1-\ln\overset{0}{\lambda})\{n^2(1+\alpha)\overset{0}{u}_n-\frac{1}{2}\frac{d^2\overset{0}{u}_n}{dx^2}+\frac{1+2\alpha}{2}n\frac{d\overset{0}{v}_n}{dx}$$

$$+n(1+\alpha)\overset{0}{w}_n\}=0,$$

$$\frac{n(1+2\alpha)}{2}(1-\ln\overset{0}{\lambda})\frac{d\overset{0}{u}_n}{dx}-\frac{n^2}{2}(1+\ln\overset{0}{\lambda})\overset{0}{v}_n+(1+\alpha)$$

$$(1-\nu\ln\overset{0}{\lambda})\frac{d^2\overset{0}{v}_n}{dx^2}+\{\alpha-(1+\alpha)\ln\overset{0}{\lambda}\}\frac{d\overset{0}{w}_n}{dx}=0,$$

$$(1+\alpha)(1-\ln\overset{0}{\lambda})n\overset{0}{u}_n+\{\alpha-(1+\alpha)\ln\overset{0}{\lambda}\}\frac{\partial\overset{0}{v}_n}{dx}+(1+\alpha)$$

$$\{n^2\ln\overset{0}{\lambda}+(1-2\ln\overset{0}{\lambda})\}\overset{0}{w}_n-\alpha(\ln\overset{0}{\lambda})\frac{d^2\overset{0}{w}_n}{dx^2}=0, \quad (26)$$

Boundary conditions are discussed as follows.

Constant Axial Displacement Condition

It is assumed that there is no difference of the axial displacement between two possible solutions at the onset of the instability, which also ensures the prevention of the rigid body motion, possibly. Under this assumption, we obtain the following condition,

$$v_z=0 \quad \text{at } z=0,\ell.$$

No Shear Traction Condition

The traction increment is,

$$\Delta T^j=L^{ijkl}\Delta u_{l,k}n_i.$$

There is no shear traction on the two free ends, i.e.,

$$\Delta\dot{T}^1=\Delta\dot{T}^3=0 \quad \text{at } z=0,\ \ell,$$

from which we may have the following condition, in terms of physical components,

$$\partial v_\theta/\partial z=\partial v_r/\partial z=0 \text{ at } z=0,\ \ell.$$

Finally, we obtain the following boundary condition

$$v_z=\partial v_\theta/\partial z=\partial v_r/\partial z=0 \text{ at } z=0,\ \ell, \quad (27)$$

or, $\overset{0}{v}_n=\partial\overset{0}{u}_n/\partial x=\partial\overset{0}{w}_n/\partial x=0,$

at $x=0,\ x_0(\equiv\ell/r_i)$. (28)

Hence, we are ready to obtain the solution of the coupled ordinary differential Eq. (26) in a straightforward manner. The solution may be the following form,

$$\overset{0}{u}_n(x)=\overset{0}{u}e^{rx},$$
$$\overset{0}{v}_n(x)=\overset{0}{v}e^{rx},$$
$$\overset{\infty n}{w}(x)=\overset{\infty}{w}e^{rx}. \quad (29)$$

Inserting (29) into (26), we get the following equations,

$$\{(1+\alpha)n^2-r^2/2\}(1-\ln\overset{0}{\lambda})\overset{0}{u}+(1/2)(1+2\alpha)nr$$
$$(1-\ln\overset{0}{\lambda})\overset{0}{v}+(1+\alpha)n(1-\ln\overset{0}{\lambda})\overset{0}{w}=0,$$
$$(1/2)(1+2\alpha)nr(1-\ln\overset{0}{\lambda})\overset{0}{u}+\{-(1/2)n^2(1+\ln\overset{0}{\lambda})$$
$$+(1+\alpha)r^2(1-\nu\ln\overset{0}{\lambda})\}\overset{0}{v}+\{\alpha-(1+\alpha)\ln\overset{0}{\lambda}\}r\overset{0}{w}=0,$$
$$(1+\alpha)n(1-\ln\overset{0}{\lambda})\overset{0}{u}+\{\alpha-(1+\alpha)\ln\overset{0}{\lambda}\}r\overset{0}{v}+[(1+\alpha)$$
$$\{n^2\ln\overset{0}{\lambda}+(1-2\ln\overset{0}{\lambda})\}-\alpha r^2\ln\overset{0}{\lambda}]\overset{0}{w}=0. \quad (30)$$

Hence, we should have the following characteristic equation for the nontrivial solution, i. e.,

$$Ar^6+Br^4+Cr^2+D=0, \quad (31)$$

with definitions

$$A=\alpha(1+\alpha)(\ln\overset{0}{\lambda})(1-\ln\overset{0}{\lambda})(1-\nu\ln\overset{0}{\lambda}),$$
$$B=(1-\ln\overset{0}{\lambda})[\{-(1+\alpha)(1+3\alpha)+\alpha^2\ln\overset{0}{\lambda}\}n^2\ln\overset{0}{\lambda}$$
$$-(1+2\alpha)+(1+\alpha)(2+\alpha)\ln\overset{0}{\lambda}$$
$$+(1+\alpha)(1-\alpha)(\ln\overset{0}{\lambda})^2],$$
$$C=n^2(1+\alpha)(\ln\overset{0}{\lambda})(1-\ln\overset{0}{\lambda})[n^2\{2+3\alpha+$$
$$(1+\alpha)\ln\overset{0}{\lambda}\}+3+2\alpha+21n\overset{0}{\lambda}],$$
$$D=-n^4(n^2-1)(1+\alpha)^2(\ln\overset{0}{\lambda})(1-\ln\overset{0}{\lambda})(1+\ln\lambda).$$

Or, defining $t\equiv r^2$, we have three roots of t, i, e., t_0, t_1, t_2 or six roots of r, i. e., $r=\pm\sqrt{t_0}$, $\pm\sqrt{t_1}$, $\pm\sqrt{t_2}$. Therefore, the corresponding solutions are

$$\overset{0}{u}_n(x)=\overset{0}{u}_0(A_0\cosh\sqrt{t_0}x+B_0\sinh\sqrt{t_0}x)$$
$$+\overset{0}{u}_1(A_1\cosh\sqrt{t_1}x+B_1\sinh\sqrt{t_1}x)$$
$$+\overset{0}{u}_2(A_2\cosh\sqrt{t_2}x+B_2\sinh\sqrt{t_2}x),$$
$$\overset{0}{v}_n(x)=\overset{0}{v}_0(A_0\sinh\sqrt{t_0}x+B_0\cosh\sqrt{t_0}x)$$
$$+\overset{0}{v}_1(A_1\sinh\sqrt{t_1}x+B_1\cosh\sqrt{t_1}x)$$
$$+\overset{0}{v}_2(A_2\sinh\sqrt{t_2}x+B_2\cosh\sqrt{t_2}x),$$
$$\overset{0}{w}_n(x)=A_0\cosh\sqrt{t_0}x+B_0\sinh\sqrt{t_0}x)$$
$$+A_1\cosh\sqrt{t_1}x+B_1\sinh\sqrt{t_1}x$$
$$+A_2\cosh\sqrt{t_2}x+B_2\sinh\sqrt{t_2}x. \quad (32)$$

Now we apply the boundary condition (28) to get the unknwon constants (A_0, A_1, A_2, B_0, B_1, B_2). Taking a proper operation, we may get following values,

$$A_1=A_2=B_0=B_1=B_2=0,\ \sinh\sqrt{t_0}x_0=0$$

$$\text{with}\quad A_0\neq 0\ \text{(arbitrary)}. \tag{33}$$

Or, equivalently, we may obtain the following constraint,

$$\sqrt{t_0}x_0=m\pi i\ \ (m=0,\ 1,\ 2,\ 3,\ \cdots,\ \text{integer}). \tag{34}$$

Or, defining $\omega\equiv m\pi/x_0$, we get

$$t_0=-\omega^2. \tag{35}$$

Hence, finally, we get the following complete solution,

$$\begin{aligned}\overset{0}{v}_\theta(\theta,\ x)&=\overset{0}{V}_\theta\sin n\theta\ \cos\omega x,\\ \overset{0}{v}_z(\theta,\ x)&=\overset{0}{V}_z\cos n\theta\sin\omega x,\\ \overset{0}{v}_r(\theta,\ x)&=\overset{0}{V}_r\cos n\theta\ \cos\omega x,\end{aligned} \tag{36}$$

where $n(\neq 0)$ is the angular wavenumber and $\omega\equiv m\pi r_i/\ell$ (m is the axial wavenumber, ℓ=cylinder length). Now, we are ready to obtain the characteristic equation governing $\overset{0}{\lambda}$. Noting that $\sqrt{t_0}=\omega i$ is one of roots of the Eq. (31), we get the following equation,

$$-A\omega^6+B\omega^4-C\omega^2+D=0.$$

Or, we may get the following equation,

$$\begin{aligned}&(1-\ln\overset{0}{\lambda})\{(1+\alpha)n^2+\frac{\omega^2}{2}\}\{\frac{1}{2}(1+\ln\overset{0}{\lambda})n^2\\ &+(1+\alpha)(1-\nu\ln\overset{0}{\lambda})\omega^2\}[\{(1+\alpha)n^2+\alpha\omega^2\}\\ &(\ln\overset{0}{\lambda})+(1+\alpha)(1-2\ln\overset{0}{\lambda})]+(1+2\alpha)(1-\ln\overset{0}{\lambda})\\ &\{\alpha-(1+\alpha)\ln\overset{0}{\lambda}\}(1+\alpha)(1-\ln\overset{0}{\lambda})n^2\omega^2-\{\alpha-\\ &(1+\alpha)\ln\overset{0}{\lambda}\}^2(1-\ln\overset{0}{\lambda})\{(1+\alpha)n^2+\frac{\omega^2}{2}\}\omega^2\\ &-\frac{(1+2\alpha)^2}{4}(1-\ln\overset{0}{\lambda})^2[\{(1+\alpha)n^2+\alpha\omega^2\}\\ &(\ln\overset{0}{\lambda})+(1+\alpha)(1-2\ln\overset{0}{\lambda})]n^2\omega^2-(1+\alpha)^2\\ &(1-\ln\overset{0}{\lambda})^2\{\frac{1}{2}(1+\ln\overset{0}{\lambda})n^2+(1+\alpha)\\ &(1-\nu\ln\overset{0}{\lambda})\omega^2\}n^2=0.\end{aligned} \tag{37}$$

Obviously, $\overset{0}{\lambda}$ has its maximum (and so its corresponding load is minimum) at $\omega=0$, from the Eq. (37). And unknown constants ($\overset{0}{V}_\theta$, $\overset{0}{V}_z$, $\overset{0}{V}_r$) should satisfy the following equation,

$$\begin{aligned}&\{(1+\alpha)n^2+\frac{\omega^2}{2}\}(1-\ln\overset{0}{\lambda})\overset{0}{V}_\theta+\frac{1+2\alpha}{2}\\ &(1-\ln\overset{0}{\lambda})n\omega\overset{0}{V}_z+(1+\alpha)(1-\ln\overset{0}{\lambda})n\overset{0}{V}_r=0,\\ &\frac{1+2\alpha}{2}(1-\ln\overset{0}{\lambda})n\omega\overset{0}{V}_\theta+\{\frac{1}{2}(1+\ln\overset{0}{\lambda})n^2-(1+\alpha)\\ &(1-\nu\ln\overset{0}{\lambda})\omega^2\}\overset{0}{V}_z+\{\alpha-(1+\alpha)\ln\overset{0}{\lambda}\}\omega\overset{0}{V}_r=0,\\ &(1+\alpha)(1-\ln\overset{0}{\lambda})n\overset{0}{V}_\theta+\{\alpha-(1+\alpha)\ln\overset{0}{\lambda}\}\omega\overset{0}{V}_z\\ &+\{(1+\alpha)(\ln\overset{0}{\lambda})n^2+\alpha(\ln\overset{0}{\lambda})\omega^2+(1+\alpha)(1-2\ln\overset{0}{\lambda}\}\\ &\overset{0}{V}_r=0.\end{aligned} \tag{38}$$

Now that the load parameter λ is a functi on of ω, we have the following general criticality feature,

$$\begin{aligned}\overset{0}{\lambda}_{cr}&=\overset{0}{\lambda}(\overset{0}{\omega}_{cr},\ n_{cr}),\\ \overset{1}{\lambda}_{cr}&=\overset{1}{\lambda}(\overset{0}{\omega}_{cr},\ n_{cr})+\frac{\partial\overset{0}{\lambda}}{\partial\omega}(\overset{0}{\omega}_{cr},\ n_{cr})\overset{1}{\omega}_{cr},\\ \overset{2}{\lambda}_{cr}&=\overset{2}{\lambda}(\overset{0}{\omega}_{cr},\ n_{cr})+\frac{\partial\overset{1}{\lambda}}{\partial\omega}(\overset{0}{\omega}_{cr},\ n_{cr})\overset{1}{\omega}_{cr}+\frac{1}{2}\\ &\quad\frac{\partial^2\overset{0}{\lambda}}{\partial\omega^2}(\overset{0}{\omega}_{cr},\ n_{cr})\overset{1}{\omega}{}^2_{cr}+\frac{\partial\overset{0}{\lambda}}{\partial\omega}(\overset{0}{\omega}_{cr},\ n_{cr})\overset{2}{\omega}_{cr},\\ \overset{3}{\lambda}_{cr}&=\overset{3}{\lambda}(\overset{0}{\omega}_{cr},\ n_{cr})+\frac{\partial\overset{2}{\lambda}}{\partial\omega}(\overset{0}{\omega}_{cr},\ n_{cr})\overset{1}{\omega}_{cr}+\frac{1}{2}\\ &\quad\frac{\partial^2\overset{1}{\lambda}}{\partial\omega^2}(\overset{0}{\omega}_{cr},n_{cr})\overset{1}{\omega}{}^2_{cr}+\frac{\partial\overset{1}{\lambda}}{\partial\omega}(\overset{0}{\omega}_{cr},\ n_{cr})\overset{2}{\omega}_{cr}\\ &\quad+\frac{1}{6}\frac{\partial^3\overset{0}{\lambda}}{\partial\omega^3}(\overset{0}{\omega}_{cr},n_{cr})\overset{1}{\omega}{}^3_{cr}+\frac{\partial^2\overset{0}{\lambda}}{\partial\omega^2}(\overset{0}{\omega}_{cr},n_{cr})\\ &\quad\overset{1}{\omega}_{cr}\overset{2}{\omega}_{cr}+\frac{\partial\overset{0}{\lambda}}{\partial\omega}(\overset{0}{\omega}_{cr},\ n_{cr})\overset{3}{\omega}_{cr}\end{aligned} \tag{39}$$

etc., with

$$\begin{aligned}&\frac{\partial\overset{0}{\lambda}}{\partial\omega}(\overset{0}{\omega}_{cr},\ n_{cr})=0,\\ &\overset{1}{\omega}_{cr}=-\frac{\partial\overset{1}{\lambda}/\partial\omega}{\partial^2\overset{0}{\lambda}/\partial\omega^2}\ \text{at}\ \omega=\overset{0}{\omega}_{cr},\ n=n_{cr},\\ &\overset{2}{\omega}_{cr}=-(\frac{\partial\overset{2}{\lambda}}{\partial\omega}+\frac{\partial^2\overset{1}{\lambda}}{\partial\omega^2}\overset{1}{\omega}_{cr}+\frac{\partial^3\overset{0}{\lambda}}{\partial\omega^3}\overset{1}{\omega}_{cr}\overset{2}{\omega}_{cr})/\frac{\partial^2\overset{0}{\lambda}}{\partial\omega^2}\\ &\quad\text{at}\ \omega=\overset{0}{\omega}_{cr},n=n_{cr},\end{aligned} \tag{40}$$

etc.

Obviously, $\overset{0}{\omega}_{cr}=0$ from the characteristic Eq. (37), and the corresponding lowest order critical load parameter and wavenumber are

$$\ln\overset{0}{\lambda}_{cr}=0\ \text{or}\ \overset{0}{\lambda}_{cr}=1$$

for $n\neq 0$, $n\neq 1$, all ν and $\overset{0}{\omega}_{cr}=0$, (41)

and the corresponding critical load is

$$(\overset{0}{p}_o)_{cr}=-\frac{E}{(1-\nu^2)\overset{0}{\lambda}_{cr}^{\frac{1-2\nu}{1-\nu}}}\ln\overset{0}{\lambda}_{cr}=0.$$

Now, to obtain the corresponding solution, we define

$$[A]=\begin{bmatrix}\overset{0}{G}{}^{<1111>}n^2+\overset{0}{G}{}^{<2121>}\omega^2, (\overset{0}{G}{}^{<1122>}+\overset{0}{G}{}^{<2112>})\,n\omega,\\ \overset{0}{G}{}^{<1111>}n, \qquad (\overset{0}{G}{}^{<1122>}+\overset{0}{G}{}^{<2112>})\,n\omega,\\ \overset{0}{G}{}^{<1212>}n^2+\overset{0}{G}{}^{<2222>}\omega^2, (\overset{0}{G}{}^{<2211>}-\overset{0}{F}{}_1^{<2332>})\,\omega,\\ \overset{0}{G}{}^{<1111>}n, \qquad (\overset{0}{G}{}^{<2211>}-\overset{0}{F}{}_1^{<2332>})\,\omega,\\ \overset{0}{G}{}^{<1313>}n^2+\overset{0}{G}{}^{<2323>}\omega^2+\overset{0}{G}{}^{<1111>}+\overset{0}{F}{}_1^{<1133>}\end{bmatrix} \tag{42}$$

where we defined the asymptotic physical components of the moduli as

$$\overset{0}{G}{}^{<ijkl>}\equiv\overset{0}{G}{}^{ijkl}\sqrt{\overset{0}{g}_{ii}\overset{0}{g}_{jj}\overset{0}{g}_{kk}\overset{0}{g}_{ll}}\ \text{(no sum)},$$

$$\overset{1}{F}{}_m^{<ijkl>}\equiv\overset{1}{F}{}_m^{ijkl}\sqrt{\overset{0}{g}_{ii}\overset{0}{g}_{jj}\overset{0}{g}_{kk}\overset{0}{g}_{ll}}\ \text{(no sum)},$$

with $\overset{0}{g}_{11}=r_i^2$, $\overset{0}{g}_{22}=\overset{0}{g}_{33}=1$ and others are zero.

And then, the Eq. (38) may take the following form

$$[A]\{\overset{0}{V}\}=\{0\}\ \text{with}\ \{\overset{0}{V}\}=\begin{bmatrix}\overset{0}{V}_\theta\\ \overset{0}{V}_z\\ \overset{0}{V}_r\end{bmatrix}, \tag{43}$$

from which we obtain the following solution

$$\overset{0}{V}_\theta=-\frac{n_{cr}\{n_{cr}^2+(2+\nu)\overset{0}{\omega}{}_{cr}^2\}}{(n_{cr}^2+\overset{0}{\omega}{}_{cr}^2)^2}\overset{0}{V}_r,$$

$$\overset{0}{V}_z=\frac{\overset{0}{\omega}_{cr}(n_{cr}^2-\nu\overset{0}{\omega}{}_{cr}^2)}{(n_{cr}^2+\overset{0}{\omega}{}_{cr}^2)^2}\overset{0}{V}_r\ \text{with}\ \overset{0}{V}_r\equiv 1,$$

or

$$\overset{0}{V}_\theta=-\frac{1}{n_{cr}}\overset{0}{V}_r,\ \overset{0}{V}_z=0,\ \overset{0}{V}_r=1\ \text{for}\ \overset{0}{\omega}_{cr}=0. \tag{44}$$

The first order analysis is as follows.

The general first order solution is, for the constant current thickness structure,

$$\int^{1/2}\{\overset{0}{G}{}^{\alpha j\beta l}\overset{1}{u}_{l,\beta}+\overset{1}{G}{}^{\alpha j\beta l}\overset{0}{u}_{l,\beta}),_{\alpha}^{0}+(\overset{1}{\Gamma}{}^{\alpha}_{m\alpha}\overset{0}{G}{}^{mj\beta l}+\overset{1}{\Gamma}{}^{j}_{m\alpha}\overset{0}{G}{}^{\alpha m\beta l}+\overset{0}{\Gamma}{}^{j}_{m3}\overset{1}{U}{}^{3m\beta l})\overset{0}{u}_{l,\beta}\}d\xi-(\overset{0}{F}{}_3^{3j\beta l}\overset{1}{w}_{l,\beta}$$
$$+\overset{0}{F}{}_2^{3j\beta l}\overset{1}{B}_{l,\beta}+\overset{1}{\bar{F}}{}_2^{3j\beta l}\overset{0}{u}_{l,\beta})=0, \tag{45}$$

with

$$\overset{1}{G}{}^{ijkl}=\overset{0}{L}{}^{ijkl}-\overset{0}{L}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{1}{L}{}^{3nkl}-\overset{1}{L}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{0}{L}{}^{3nkl}$$
$$+\overset{0}{L}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{1}{L}{}^{3n3p}(\overset{0}{L}{}^{3q3p})^{-1}\overset{0}{L}{}^{3qkl}$$
$$\equiv\overset{1}{\bar{X}}{}^{ijkl}\xi+\overset{1}{\bar{Y}}{}^{ijpq},$$
$$\overset{1}{U}{}^{ijkl}=\overset{1}{L}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{0}{F}{}^{3nkl},$$
$$\overset{1}{G}{}^{ijkl}=\overset{1}{G}{}^{ijkl}+\overset{1}{U}{}^{ijkl}\equiv\overset{1}{\bar{X}}{}^{ijpq}\xi+\overset{1}{\bar{Y}}{}^{ijpq},$$

and

$$\overset{1}{F}{}^{ijkl}=\overset{2}{N}{}^{ijkl}-\overset{1}{N}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{1}{L}{}^{3nkl}+\overset{1}{N}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}$$
$$\overset{1}{L}{}^{3n3p}(\overset{0}{L}{}^{3q3p})^{-1}\overset{0}{L}{}^{3qkl}-\overset{2}{N}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{0}{L}{}^{3nkl}$$
$$\equiv\overset{1}{F}{}_1^{ijkl}\xi^2+\overset{1}{F}{}_2^{ijkl}\xi+\overset{1}{F}{}_3^{ijkl},$$
$$\overset{1}{V}{}^{ijkl}=-\overset{1}{N}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{0}{F}{}^{3nkl},$$
$$\overset{1}{\bar{F}}{}^{ijkl}=\overset{1}{F}{}^{ijkl}+\overset{1}{V}{}^{ijkl}\equiv\overset{1}{\bar{F}}{}_1^{ijkl}\xi^2+\overset{1}{\bar{F}}{}_2^{ijkl}\xi+\overset{1}{\bar{F}}{}_3^{ijkl},$$

and

$$\frac{\partial\overset{1}{u}_l}{\partial\xi}=-(\overset{0}{L}{}^{3j3l})^{-1}(\overset{0}{L}{}^{3j\beta k}\overset{0}{u}_{k,\beta}-\overset{0}{L}{}^{3j3k}\overset{0}{u}_n\overset{0}{\Gamma}{}^n_{k3})\equiv\overset{1}{B}_l$$

or

$$\overset{1}{u}_l=\overset{1}{B}_l\xi+\overset{1}{w}_l(\theta^\alpha). \tag{46}$$

Now we define the asymptotic physical components as

$$\overset{1}{w}_1=r_i\overset{1}{w}_\theta,\ \overset{1}{w}_2=\overset{1}{w}_z,\ \overset{1}{w}_3=\overset{1}{w}_r.$$

And then, using the updated moduli, the solution (46) may be

$$\overset{1}{u}_1=r_i\{-(\frac{\partial\overset{0}{v}_r}{\partial\theta}-2\overset{0}{v}_\theta)\xi+\overset{1}{w}_\theta\},$$
$$\overset{1}{u}_2=-\frac{\partial\overset{0}{v}_r}{\partial x}\xi+\overset{1}{w}_z,$$
$$\overset{1}{u}_3=-\alpha(\frac{\partial\overset{0}{v}_\theta}{\partial\theta}+\frac{\partial\overset{0}{v}_z}{\partial x}+\overset{0}{v}_r)\xi+\overset{1}{w}_r. \tag{47}$$

The general solution (45) becomes the following coupled partial differential equations governing the unknown midsurface values in (47), after quite a long algebra,

$$\overset{0}{G}{}^{<1111>}\frac{\partial^2\overset{1}{w}_\theta}{\partial\theta^2}+\overset{0}{G}{}^{<2121>}\frac{\partial^2\overset{1}{w}_\theta}{\partial x^2}+(\overset{0}{G}{}^{<1122>}+\overset{0}{G}{}^{<2112>})$$

$$\frac{\partial^2 \overset{1}{w}_z}{\partial\theta\partial x}+\overset{0}{G}{}^{<1111>}\frac{\partial \overset{1}{w}_r}{\partial\theta}=-g_1\sin n\theta\cos\omega x,$$

$$(\overset{0}{G}{}^{<1221>}+\overset{0}{G}{}^{<2211>})\frac{\partial^2\overset{1}{w}_\theta}{\partial\theta\partial x}+\overset{0}{G}{}^{<1212>}\frac{\partial^2\overset{1}{w}_z}{\partial\theta^2}+\overset{0}{G}{}^{<2222>}\frac{\partial^2\overset{1}{w}_z}{\partial x^2}-(\overset{0}{G}{}^{<2211>}-\overset{0}{F}{}_1^{<3223>})\frac{\partial\overset{1}{w}_r}{\partial x}=-g_2\sin n\theta\cos\omega x,$$

$$\overset{0}{G}{}^{<1111>}\frac{\partial\overset{1}{w}_\theta}{\partial\theta}+(\overset{0}{G}{}^{<1122>}+\overset{0}{F}{}_1^{<3322>})\frac{\partial\overset{1}{w}_z}{\partial x}-\overset{0}{G}{}^{<1313>}\frac{\partial^2\overset{1}{w}_r}{\partial\theta^2}-\overset{0}{G}{}^{<2323>}\frac{\partial^2\overset{1}{w}_r}{\partial x^2}+(\overset{0}{G}{}^{<1111>}+\overset{0}{F}{}_1^{<3311>})\overset{1}{w}_r$$
$$=-g_3\cos n\theta\cos\omega x, \qquad (48)$$

where g_1, g_2, g_3 are terms due to the lowest order solution and have long expressions (and so their explicit forms are not recorded) . Now that the above coupled partial differential equations are not homogeneous, we should note that the complete solution consists of the homogeneous solution and the particular solution. Now, the homogeneous solution must be exactly same as the lowest order solution, since the solution of the homogeneous linear boundary value problem is unique. Hence, we may obtain the following first order solution,

$$\begin{aligned}\overset{1}{w}_\theta(\theta,\ x)&=\overset{1}{W}_\theta\sin n\theta\cos\omega x,\\ \overset{1}{w}_z(\theta,\ x)&=\overset{1}{W}_z\cos n\theta\sin\omega x,\\ \overset{1}{w}_r(\theta,\ x)&=\overset{1}{W}_r\cos n\theta\cos\omega x,\end{aligned} \qquad (49)$$

with

$$\overset{1}{W}_\theta=\overset{0}{V}_\theta+\overset{1}{W}_{\theta p},\ \overset{1}{W}_z=\overset{0}{V}_z+\overset{1}{W}_{zp},\ \overset{1}{W}_r=\overset{0}{V}_r+\overset{1}{W}_{rp}, \qquad (50)$$

where the subscript p means the particular solution. And the particular solution should satisfy the following equation

$$[A]\{\overset{1}{W}_p\}=\{g\}\ \text{with}\ \{\overset{1}{W}_p\}=\begin{bmatrix}\overset{1}{W}_{\theta p}\\ \overset{1}{W}_{zp}\\ \overset{1}{W}_{rp}\end{bmatrix},\ \{g\}=\begin{bmatrix}g_1\\ g_2\\ g_3\end{bmatrix}, \qquad (51)$$

Taking a proper operation, we may obtain the following equation,

$$\{\overset{1}{W}_p\}^T[A]\{\overset{0}{V}\}=\{\overset{1}{W}_p\}^T\{0\}=0=\{g\}^T\{\overset{0}{V}\},$$

noting $[A]^T=[A]$. That is, we get

$$\overset{0}{V}_\theta g_1+\overset{0}{V}_z g_2+\overset{0}{V}_r g_3=0, \qquad (52)$$

which is the desired $\overset{1}{\lambda}$ equation.

Now using updated modui and the lowest order solution, g_1, g_2, g_3 take the following forms

$$g_1=-\frac{n\overset{0}{V}_r\overset{0}{G}{}^{<1111>}}{2(n^2+\omega^2)^2}\{n^4+2(1+\nu)n^2\omega^2-\omega^4\},$$

$$g_2=-\frac{\omega\overset{0}{V}_r\overset{0}{G}{}^{<1111>}}{2(n^2+\omega^2)^2}[\{\nu+\frac{-2-\nu+4\nu^2+3\nu^3}{2(1-\nu)}\overset{1}{\lambda}\}n^4$$
$$+\{2(1+\nu)+\frac{-8+11\nu^2+6\nu^3+3\nu^4}{2(1-\nu)}\overset{1}{\lambda}\}n^2\omega^2$$
$$-\{\nu+2(1-\nu^2)^1\}\omega^4],$$

$$g_3=-\frac{\overset{0}{V}_r\overset{0}{G}{}^{<1111>}}{2(n^2+\omega^2)^2}[\{1+2(n^2-1+\nu\omega^2)\overset{1}{\lambda}\}n^4$$
$$+\{2(1+\nu)+2(-3+\nu+2n^2+2\nu\omega^2)\overset{1}{\lambda}\}n^2\omega^2$$
$$+\{-2+\nu^2+2(-3+2\nu+n^2+\nu\omega^2)\overset{1}{\lambda}\}\omega^4]. \qquad (53)$$

Hence, we may obtain the general expression of $\overset{1}{\lambda}$ as,

$$\overset{1}{\lambda}=-2(1-\nu^2)\omega^4(n^4-\omega^4)/[2(n^2-1+\nu\omega^2)n^8$$
$$+\{\frac{-6+7\nu+3\nu^3}{2(1-\nu)}+8(n^2-1+\nu\omega^2)\}n^6\omega^2$$
$$+\{\frac{-12+17\nu-2\nu^2+\nu^3}{1-\nu}$$
$$+12(n^2-1+\nu\omega^2)\}n^4\omega^4$$
$$+\{\frac{-24+52\nu-16\nu^2-15\nu^3-6\nu^4-3\nu^5}{2(1-\nu)}$$
$$+8(n^2-1+\nu\omega^2)\}n\omega^6+\{-2(1-\nu)^2(2+\nu)$$
$$+2(n^2-1+\nu\omega^2)\}\omega^8]+(\ln\overset{0}{\lambda})\overset{0}{f}(\omega^2,n^2,\nu). \qquad (54)$$

We can notice that $\overset{1}{\lambda}$ is an even function of ω, n. Now from the general criticality features (39) and (40), and using the lowest order criticality (41), we can deduce the following first order criticality

$$\overset{1}{\lambda}_{cr}=\overset{1}{\lambda}(\overset{0}{\omega}_{cr},\ n_{cr})=0,$$
$$\overset{1}{\omega}_{cr}=-\frac{\partial\overset{1}{\lambda}/\partial\omega}{\partial^2\overset{0}{\lambda}/\partial\omega^2}(\overset{0}{\omega}_{cr},\ n_{cr})=0, \qquad (55)$$

and the corresponding critical load is

$$(\overset{2}{p}_o)_{cr}=-\frac{E}{1-\nu^2}\overset{1}{\lambda}_{cr}=0. \qquad (56)$$

And the corresponding critical mode may be obtained, using updated moduli and the mode orthogonality condition, as

$$\overset{1}{W}_\theta=-\frac{1}{2n_{cr}}\overset{0}{V}_r,\ \overset{1}{W}_z=0,\ \overset{1}{W}_r=0. \tag{57}$$

The second order analysis is as follows.

The second order general solution is, for the constant current thickness,

$$\begin{aligned}
&\int_{-1/2}^{1/2}\{(\overset{0}{G}{}^{\alpha j\beta l}\overset{2}{u}_{l,\beta}+\overset{1}{G}{}^{\alpha j\beta l}\overset{1}{u}_{l,\beta}+\overset{2}{G}{}^{\alpha j\beta l}\overset{0}{u}_{l,\beta}),\overset{0}{{}_\alpha}\\
&+\overset{1}{\Gamma}{}^{\alpha}_{ma}(\overset{0}{G}{}^{mj\beta l}\overset{1}{u}_{l,\beta}+\overset{1}{G}{}^{mj\beta l}\overset{0}{u}_{l,\beta})+\overset{1}{\Gamma}{}^{j}_{ma}(\overset{0}{G}{}^{\alpha m\beta l}\overset{1}{u}_{l,\beta}\\
&+\overset{1}{G}{}^{\alpha m\beta l}\overset{0}{u}_{l,\beta})+\overset{2}{\Gamma}{}^{\alpha}_{ma}\overset{0}{G}{}^{mj\beta l}\overset{0}{u}_{l,\beta}\\
&+\overset{2}{\Gamma}{}^{j}_{ma}\overset{0}{G}{}^{\alpha m\beta l}\overset{0}{u}_{l,\beta}\}d\xi-\{\overset{0}{F}{}^{3j\beta l}_{1}\overset{2}{w}_{l,\beta}+\overset{0}{F}{}^{3j\beta l}_{2}\overset{2}{D}_{l,\beta}\\
&+\overset{1}{F}{}^{3j\beta l}_{3}\overset{1}{B}_{l,\beta}+\overset{1}{F}{}^{3j\beta l}_{2}\overset{1}{w}_{l,\beta}+\overset{2}{F}{}^{3j\beta l}_{3}\overset{0}{u}_{l,\beta}\\
&+\frac{1}{4}(\frac{1}{2}\overset{0}{F}{}^{3j\beta l}_{1}\overset{2}{B}_{l,\beta}\overset{1}{F}{}^{3j\beta l}_{1}\overset{1}{B}_{l,\beta}+\overset{2}{F}{}^{3j\beta l}_{1}\overset{0}{u}_{l,\beta})\}\\
&=\int_{-1/2}^{1/2}[\{\overset{0}{L}{}^{\alpha j3k}(\overset{0}{L}{}^{3m3k})^{-1}\overset{1}{a}{}^{m}\},\overset{0}{{}_\alpha}+\overset{0}{\Gamma}{}^{j}_{m3}\overset{1}{a}{}^{m}\\
&-\overset{1}{\Gamma}{}^{j}_{m3}\overset{1}{U}{}^{3m\beta l}\overset{0}{u}_{l,\beta}]d\xi-\{\overset{1}{A}{}^{3j3k}(\overset{0}{L}{}^{3m3k})^{-1}\\
&(\frac{1}{4}\overset{1}{\alpha}{}^{m}\overset{1}{\gamma}{}^{m})+\overset{1}{B}{}^{3j3k}(\overset{0}{L}{}^{3m3k})^{-1}\overset{1}{\beta}{}^{m}\},
\end{aligned}\tag{58}$$

with

$$\begin{aligned}
\overset{2}{G}{}^{ijkl}=&\overset{2}{L}{}^{ijkl}-\overset{0}{L}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{2}{L}{}^{3nkl}+\overset{0}{L}{}^{ij3m}\\
&(\overset{0}{L}{}^{3n3m})^{-1}\overset{1}{L}{}^{3n3t}(\overset{0}{L}{}^{3p3t})^{-1}\overset{1}{L}{}^{3pkl}-\overset{0}{L}{}^{ij3m}\\
&(\overset{0}{L}{}^{3n3m})^{-1}\overset{1}{L}{}^{3n3t}(\overset{0}{L}{}^{3p3t})^{-1}\overset{1}{L}{}^{3p3q}(\overset{0}{L}{}^{3r3q})^{-1}\\
&\overset{0}{L}{}^{3rkl}+\overset{0}{L}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{2}{L}{}^{3n3t}(\overset{0}{L}{}^{3p3t})^{-1}\\
&\overset{0}{L}{}^{3pkl}-\overset{1}{L}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{1}{L}{}^{3nkl}+\overset{1}{L}{}^{ij3m}\\
&(\overset{0}{L}{}^{3n3m})^{-1}\overset{1}{L}{}^{3n3p}(\overset{0}{L}{}^{3r3p})^{-1}\overset{0}{L}{}^{3rkl}-\overset{2}{L}{}^{ij3m}\\
&(\overset{0}{L}{}^{3n3m})^{-1}\overset{0}{L}{}^{3nkl},
\end{aligned}$$

$$\overset{2}{U}{}^{ijkl}=-\{\overset{1}{L}{}^{ij3t}-\overset{0}{L}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{1}{L}{}^{3n3t}\}(\overset{0}{L}{}^{3p3t})^{-1}\overset{1}{F}{}^{3pkl},$$

$$\overset{2}{\bar{G}}{}^{ijkl}=\overset{2}{G}{}^{ijkl}+\overset{2}{U}{}^{ijkl},$$

and

$$\begin{aligned}
\overset{2}{F}{}^{ijkl}=&\overset{3}{N}{}^{ijkl}-\overset{1}{N}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{2}{L}{}^{3nkl}+\overset{1}{N}{}^{ij3m}\\
&(\overset{0}{L}{}^{3n3m})^{-1}\overset{1}{L}{}^{3n3p}(\overset{0}{L}{}^{3r3p})^{-1}\overset{1}{L}{}^{3rkl}-\overset{1}{N}{}^{ij3m}\\
&(\overset{0}{L}{}^{3n3m})^{-1}\overset{1}{L}{}^{3n3p}(\overset{0}{L}{}^{3r3p})^{-1}\overset{1}{L}{}^{3r3t}(\overset{0}{L}{}^{3q3t})^{-1}\\
&\overset{0}{L}{}^{3qkl}+\overset{1}{N}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{2}{L}{}^{3n3p}(\overset{0}{L}{}^{3r3p})^{-1}\\
&\overset{0}{L}{}^{3rkl}-\overset{2}{N}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{1}{L}{}^{3nkl}+\overset{2}{N}{}^{ij3m}\\
&(\overset{0}{L}{}^{3n3m})^{-1}\overset{1}{L}{}^{3n3p}(\overset{0}{L}{}^{3r3p})^{-1}\overset{0}{L}{}^{3rkl}-\overset{3}{N}{}^{ij3m}\\
&(\overset{0}{L}{}^{3n3m})^{-1}\overset{0}{L}{}^{3nkl},
\end{aligned}$$

$$\begin{aligned}
\overset{2}{V}{}^{ijkl}=&\{\overset{1}{N}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{1}{L}{}^{3n3p}-\overset{2}{N}{}^{ij3p}\}\\
&(\overset{0}{L}{}^{3t3p})^{-1}\overset{0}{F}{}^{3tkl},
\end{aligned}$$

$$\overset{2}{\bar{F}}{}^{ijkl}=\overset{2}{F}{}^{ijkl}+\overset{2}{V}{}^{ijkl}.$$

And also we have

$$\begin{aligned}
\frac{\partial\overset{2}{u}_l}{\partial\xi}=&-(\overset{0}{L}{}^{3j3l})^{-1}[\{\overset{0}{F}{}^{3j\beta k}+\overset{1}{L}{}^{3j\beta k}-\overset{1}{L}{}^{3j3n}\\
&(\overset{0}{L}{}^{3r3n})^{-1}\overset{0}{L}{}^{3r\beta k}\}\overset{0}{u}_{k,\beta}+\overset{0}{L}{}^{3j\beta k}\overset{1}{u}_{k,\beta}\\
&-\overset{0}{L}{}^{3j3k}(\overset{0}{u}_n\overset{1}{\Gamma}{}^{n}_{k3}+\overset{1}{u}_n\overset{0}{\Gamma}{}^{n}_{k3})]\equiv\overset{2}{B}_l\xi+\overset{2}{D}_l,
\end{aligned}$$

$$\text{or } \overset{2}{u}_l=\frac{1}{2}\overset{2}{B}_l\overset{2}{\xi}+\overset{2}{D}_l\xi+\overset{2}{w}_l(\theta^\alpha), \tag{59}$$

and

$$\begin{aligned}
\overset{1}{a}{}^{j}=&\int_{-1/2}^{\xi}\{(\overset{0}{G}{}^{\alpha j\beta l}\overset{1}{u}_{l,\beta}+\overset{1}{G}{}^{\alpha j\beta l}\overset{0}{u}_{l,\beta}),\overset{0}{{}_\alpha}+(\overset{1}{\Gamma}{}^{\alpha}_{ma}\overset{0}{G}{}^{mj\beta l}\\
&+\overset{1}{\Gamma}{}^{j}_{ma}\overset{0}{G}{}^{\alpha m\beta l}+\overset{0}{\Gamma}{}^{j}_{m3}\overset{1}{U}{}^{3m\beta l})\overset{0}{u}_{l,\beta}\}d\xi-\{(\frac{1}{2}\overset{0}{F}{}^{3j\beta l}_{1}\\
&-\overset{0}{F}{}^{3j\beta l}_{2})(-\frac{1}{2}\overset{1}{B}_{l,\beta}+\overset{1}{w}_{l,\beta})+(-\frac{1}{4}\overset{1}{F}{}^{3j\beta l}_{1}\\
&+\frac{1}{2}\overset{1}{F}{}^{3j\beta l}_{2}-\overset{1}{F}{}^{3j\beta l}_{3})\overset{0}{u}_{l,\beta}\}\equiv\overset{1}{\alpha}{}^{j}\xi^2+\overset{1}{\beta}{}^{j}\xi+\overset{1}{\gamma}{}^{j}.
\end{aligned}\tag{60}$$

Now using updated moduli, the solution (59) may take the following form,

$$\begin{aligned}
\overset{2}{u}_1=&r_i[\frac{1}{2}\{\frac{\nu}{1-\nu}\frac{\partial}{\partial\theta}(\frac{\partial\overset{0}{v}_\theta}{\partial\theta}+\frac{\partial\overset{0}{v}_z}{\partial x})-\frac{2-3\nu}{1-\nu}\frac{\partial\overset{0}{v}_r}{\partial\theta}\\
&+2\overset{0}{v}_\theta\}\xi^2+(-\overset{0}{v}_\theta+2\overset{1}{w}_\theta-\frac{\partial\overset{1}{w}_r}{\partial\theta})\xi+\overset{2}{w}_\theta],
\end{aligned}$$

$$\begin{aligned}
\overset{2}{u}_2=&\frac{\nu}{2(1-\nu)}\frac{\partial}{\partial x}(\frac{\partial\overset{0}{v}_\theta}{\partial\theta}+\frac{\partial\overset{0}{v}_z}{\partial x}+\overset{0}{v}_r)\xi^2\\
&-\frac{\partial\overset{1}{w}_r}{\partial x}\xi+\overset{2}{w}_z,
\end{aligned}$$

$$\begin{aligned}
\overset{2}{u}_3=&\frac{\nu}{2(1-\nu)}(\frac{\partial^2\overset{0}{v}_r}{\partial\theta^2}+\frac{\partial^2\overset{0}{v}_r}{\partial x^2}+\frac{\nu}{1-\nu}\frac{\partial\overset{0}{v}_\theta}{\partial\theta}\\
&+\frac{\nu}{1-\nu}\frac{\partial\overset{0}{v}_z}{\partial x}+\frac{1}{1-\nu}\overset{0}{v}_r)\xi^2+\frac{\nu}{1-\nu}(\frac{\partial\overset{0}{v}_\theta}{\partial\theta}\\
&+\frac{1}{2}\overset{0}{v}_r-\frac{\partial\overset{1}{w}_\theta}{\partial\theta}-\frac{\partial\overset{1}{w}_z}{\partial x}-\overset{1}{w}_r)\xi+\overset{2}{w}_r,
\end{aligned}\tag{61}$$

where we defined the asymptotic physical components of the unknown midsurface mode as

$$\overset{2}{w}_1=r_i\overset{2}{w}_\theta,\ \overset{2}{w}_2=\overset{2}{w}_z,\ \overset{2}{w}_3=\overset{2}{w}_r.$$

The general Eq. (59) becomes the following coupled partial differential equations governing the unknown midsurface second order mode in (61) as shown below,

$$-\left(\overset{0}{G}^{<1111>}\frac{\partial^2 \overset{2}{w}_\theta}{\partial\theta^2}+\overset{0}{G}^{<2121>}\frac{\partial^2\overset{2}{w}_\theta}{\partial x^2}\right)-\left(\overset{0}{G}^{<2112>}+\overset{0}{G}^{<1122>}\right)\frac{\partial^2\overset{2}{w}_z}{\partial\theta\partial x}-\overset{0}{G}^{<1111>}\frac{\partial\overset{2}{w}_r}{\partial\theta}=f_1\sin n\theta\cos\omega x,$$

$$-\left(\overset{0}{G}^{<1221>}+\overset{0}{G}^{<2211>}\right)\frac{\partial^2\overset{2}{w}_\theta}{\partial\theta\partial x}-\left(\overset{0}{G}^{<1212>}\frac{\partial^2\overset{2}{w}_z}{\partial\theta^2}+\overset{0}{G}^{<2222>}\frac{\partial^2\overset{2}{w}_z}{\partial x^2}\right)-\overset{0}{G}^{<2211>}\frac{\partial\overset{2}{w}_r}{\partial x}=f_2\cos n\theta\sin\omega x,$$

$$\overset{0}{G}^{<1111>}\frac{\partial\overset{2}{w}_\theta}{\partial\theta}+\overset{0}{G}^{<1122>}\frac{\partial\overset{2}{w}_z}{\partial x}+\overset{0}{G}^{<1111>}\overset{2}{w}_r=f_3\cos n\theta\cos\omega x, \qquad (62)$$

where f_1, f_2, f_3 usually take the long expressions (and so their tedious forms are not recorded). As discussed in the first order analysis, the general solution of the Eq. (62) is,

$$\overset{2}{w}_\theta(\theta,\ n)=\overset{2}{W}_\theta\sin n\theta\cos\omega x,$$
$$\overset{2}{w}_z(\theta,\ n)=\overset{2}{W}_z\cos n\theta\sin\omega x,$$
$$\overset{2}{w}_r(\theta,\ n)=\overset{2}{W}_r\cos n\theta\cos\omega x, \qquad (63)$$

where we may have,

$$\overset{2}{W}_\theta=\overset{0}{V}_\theta+\overset{2}{W}_{\theta p},\ \overset{2}{W}_z=\overset{0}{V}_z+\overset{2}{W}_{zp},$$
$$\overset{2}{W}_r=\overset{0}{V}_r+\overset{2}{W}_{rp}. \qquad (64)$$

The particular solution should also satisfy the following equation

$$[A]\{\overset{2}{W}_p\}=\{f\}\ \text{with}\ \{\overset{2}{W}_p\}\equiv\begin{bmatrix}\overset{2}{W}_{\theta p}\\ \overset{2}{W}_{zp}\\ \overset{2}{W}_{rp}\end{bmatrix},\ \{f\}\equiv\begin{bmatrix}f_1\\ f_2\\ f_3\end{bmatrix}. \qquad (65)$$

Also taking the same operation as in the first order analysis, we have the $\overset{2}{\lambda}$ equation as

$$\overset{0}{V}_\theta f_1+\overset{0}{V}_z f_2+\overset{0}{V}_r f_3=0. \qquad (66)$$

Now using updated moduli and the lowest order and the first order solutions, f_1, f_2, f_3 may take following forms

$$f_1=\frac{n\overset{0}{G}^{<1111>}}{24(1-\nu)(n^2+\omega^2)^2}\{3\nu n^6-3(1-4\nu)n^4\omega^2-3(2-5\nu)n^2\omega^4-3(1-2\nu)\omega^6-3\nu n^4-3(7+2\nu-6\nu^2)n^2\omega^2+(24-44\nu-3\nu^2+17\nu^3)\omega^4\}\overset{0}{V}_r+\frac{2(1-\nu)\omega^2(n^2-\nu\omega^2)}{(n^2+\omega^2)^2}\overset{0}{G}^{<1111>}\overset{2}{\lambda}n\overset{0}{V}_r,$$

$$f_2=\frac{\omega\overset{0}{G}^{<1111>}}{24(1-\nu)(n^2+\omega^2)^2}\{(1+2\nu^2)n^6+3\nu(1+1\nu)n^4\omega^2-3(1-2\nu-2\nu^2)n^2\omega^4-(2-3\nu-2\nu^2)\omega^6-(1+2\nu^2)n^4-9(2+\nu-2\nu^2)n^2\omega^2+\nu(2-21\nu+13\nu^2)\omega^4\}\overset{0}{V}_r+\frac{\overset{0}{G}^{<1111>}}{(n^2+\omega^2)^2}\{(-2+\nu)n^4+(-3+\nu^2)n^2\omega^2+(1-\nu-\nu^2)\omega^4\}\overset{2}{\lambda}\omega\overset{0}{V}_r,$$

$$f_3=\frac{\overset{0}{G}^{<1111>}}{24(1-\nu)(n^2+\omega^2)^2}\{-2(1-\nu)(n^2+\omega^2)^4+(4-\nu)n^6+(12-4\nu+\nu^2)n^4\omega^2+(8+4\nu^2-3\nu^3)n^2\omega^4+3\nu(1+\nu-\nu^2)\omega^6-(2+\nu)n^4+(-28+2\nu+17\nu^2)n^2\omega^2+(40-59\nu-25\nu^2+38\nu^3)\omega^4\}\overset{0}{V}_r+\frac{\overset{0}{G}^{<1111>}}{(n^2+\omega^2)^2}\{-(n^2+\nu\omega^2)(n^2+\omega^2)^2+n^4+(3-\nu)n^2\omega^2+(3-2\nu-2\nu^2)\omega^4\}\overset{2}{\lambda}\overset{0}{V}_r. \qquad (67)$$

Hence, we may obtain the following $\overset{2}{\lambda}$ expression, using (66) and (67),

$$\overset{2}{\lambda}=\{-2(n^2+\omega^2)^6+4n^{10}+24n^8\omega^2+(48+5\nu+5\nu^2)n^6\omega^4+(40+15\nu+15\nu^2)n^4\omega^6+(12+15\nu+15\nu^2)n^2\omega^8+5\nu(1+\nu)\omega^{10}-2n^8-12n^6\omega^2-(18+5\nu+5\nu^2)n^4\omega^4+(4-28\nu-23\nu^2+17\nu^3)n^2\omega^6+(40-19\nu-46\nu^2+13\nu^3)\omega^8\}/24\{(n^2+\nu\omega^2)(n^2+\omega^2)^4-n^8-(1+2\nu)n^6\omega^2+(-3-2\nu+2\nu^2)n^4\omega^4+(-10-\nu+7\nu^2+3\nu^3)n^2\omega^6+(1-\nu)(-3+\nu^2)\omega^8\}+(\ln\overset{0}{\lambda})\overset{1}{f}(\omega^2,\ n^2)+\overset{1}{\lambda}\overset{1}{g}(\omega^2,\ n^2). \qquad (68)$$

And so, the critical values are,

$$\overset{2}{\lambda}_{cr}=\overset{2}{\lambda}(\overset{0}{\omega}_{cr},\ n_{cr})=-\frac{n_{cr}^2-1}{12},$$
$$\overset{2}{\omega}_{cr}=0\ \text{for}\ n_{cr}\neq 0,\ n_{cr}\neq 1, \qquad (69)$$

and the corresponding critical load is

$$(\overset{3}{p}_o)_{cr}=-\frac{E}{1-\nu^2}\overset{2}{\lambda}_{cr}=\frac{E(n_{cr}^2-1)}{12(1-\nu^2)}. \qquad (70)$$

The corresponding solution is obtained as, using the mode orthogonality condition,

$$\overset{2}{W}_\theta=\frac{\alpha n_{cr}(n_{cr}^2-1)}{12(n_{cr}^2+1)}\overset{0}{V}_r,\ \overset{2}{W}_z=0,$$
$$\overset{2}{W}_r=\frac{\alpha(n_{cr}^2+3)(n_{cr}^2-1)}{24(n_{cr}^2+1)}\overset{0}{V}_r,$$

with $\alpha\equiv\frac{\nu}{1-\nu}$. (71)

The third order analysis is as follows.

The general third order solution has long tedious expressions. Hence, recording them is omitted here, and we just make sure their updated expressions. The explicit forms of the third order general solution are also quite long in the general sense and so they will not be recorded, either. But, using the updated moduli and the previous solutions, they may take the following simple forms

$$\overset{3}{u}_1=r_i[\{\frac{(2-\nu)n_{cr}(n_{cr}^2-1)}{6(1-\nu)}\xi^3-\frac{n_{cr}{}^2-1}{2n_{cr}}\xi^2-\frac{n_{cr}(n_{cr}^2-1)}{4(1-\nu)}\xi+\frac{\alpha n_{cr}(n_{cr}^2-1)(n_{cr}^2+7)}{24(n_{cr}^2+1)}\xi\}\overset{0}{V}_r\sin n\theta\cos\omega x+\overset{3}{w}_\theta],$$
$$\overset{3}{u}_2=\overset{3}{w}_z,$$
$$\overset{3}{u}_3=\{\frac{1+\nu}{6(1-\nu)}\xi^3+\frac{\nu}{2(1-\nu)}\xi^2-\frac{1}{8}\xi\}(n_{cr}^2-1)\overset{0}{V}_r\cos n\theta\cos\omega x+\overset{3}{w}_r, \quad (72)$$

where we defined the asymptotic physical components of the unknown midsurface third order mode as

$$\overset{3}{w}_1=r_i\overset{3}{w}_\theta,\ \overset{3}{w}_2=\overset{3}{w}_z,\ \overset{3}{w}_3=\overset{3}{w}_r.$$

We may also obtain the following coupled partial differential equations governing the unknown midsurface third order mode in (72) as

$$-(\overset{0}{G}{}^{<1111>}\frac{\partial^2\overset{3}{w}_\theta}{\partial\theta^2}+\overset{0}{G}{}^{<2121>}\frac{\partial^2\overset{3}{w}_\theta}{\partial x^2})-(\overset{0}{G}{}^{<2112>}+\overset{0}{G}{}^{<1122>})\frac{\partial^2\overset{3}{w}_z}{\partial\theta\partial x}-\overset{0}{G}{}^{<1111>}\frac{\partial\overset{3}{w}_r}{\partial\theta}=h_1\sin n\theta\cos\omega x,$$
$$-(\overset{0}{G}{}^{<1221>}+\overset{0}{G}{}^{<2211>})\frac{\partial^2\overset{3}{w}_\theta}{\partial\theta\partial x}-(\overset{0}{G}{}^{<1212>}\frac{\partial^2\overset{3}{w}_z}{\partial\theta^2}+\overset{0}{G}{}^{<2222>}\frac{\partial^2\overset{3}{w}_z}{\partial x^2})-\overset{0}{G}{}^{<2211>}\frac{\partial\overset{3}{w}_r}{\partial x}=h_2\cos n\theta\sin\omega x,$$
$$\overset{0}{G}{}^{<1111>}\frac{\partial\overset{3}{w}_\theta}{\partial\theta}+\overset{0}{G}{}^{<1122>}\frac{\partial\overset{3}{w}_z}{\partial x}+\overset{0}{G}{}^{<1111>}\overset{3}{w}_r=h_3\cos n\theta\cos\omega x, \quad (73)$$

where the extremely long general expressions of h_1, h_2, h_3 are not recorded here. The solution of the above equation is obviously as discussed in the first order solution, for the long cylindrical shell, i.e.,

$$\overset{3}{w}_\theta(\theta,\ n)=\overset{3}{W}_\theta\sin n\theta\cos\omega x,$$
$$\overset{3}{w}_z(\theta,\ n)=\overset{3}{W}_z\sin n\theta\cos\omega x,$$
$$\overset{3}{w}_r(\theta,\ n)=\overset{3}{W}_r\sin n\theta\cos\omega x, \quad (74)$$

where we also have

$$\overset{3}{W}_\theta=\overset{0}{V}_\theta+\overset{3}{W}_{\theta p},$$
$$\overset{3}{W}_z=\overset{0}{V}_z+\overset{3}{W}_{zp},$$
$$\overset{3}{W}_r=\overset{0}{V}_r+\overset{3}{W}_{rp}. \quad (75)$$

Thus following the same analysis adopted in the previous analyses, we have the $\overset{3}{\lambda}$ equation as

$$\overset{0}{V}_\theta h_1+\overset{0}{V}_z h_2+\overset{0}{V}_r h_3=0. \quad (76)$$

Now h_1, h_2, h_3 can be simplified greatly as, if we use the updated moduli and the previous solutions,

$$h_1=-\frac{\alpha\overset{0}{G}{}^{<1111>}n(2n^2+3)(n^2-1)}{24(n^2+1)}\overset{0}{V}_r,$$
$$h_2=0,$$
$$h_3=-\frac{\alpha\overset{0}{G}{}^{<1111>}n(2n^2+3)(n^2-1)}{24(n^2+1)}\overset{0}{V}_r-\overset{0}{G}{}^{<1111>}\overset{3}{\lambda}(n^2-1)\overset{0}{V}_r$$

noting $\alpha\equiv\frac{\nu}{1-\nu}$. (77)

Hence, finally, we may get the third order critical values as, using (76), (39)~(40),

$$\overset{3}{\lambda}_{cr}=\overset{3}{\lambda}(\overset{0}{\omega}_{cr},\ n_{cr})=0,\ \overset{3}{\omega}_{cr}=0, \quad (78)$$

and the corresponding load is

$$(\overset{4}{p}_o)_{cr}=-\frac{E}{1-\nu^2}(\overset{3}{\lambda}_{cr}-\frac{3}{2}\overset{2}{\lambda}_{cr})=-\frac{E(n_{cr}^2-1)}{8(1-\nu^2)}. \quad (79)$$

Thus, we may obtain the four term expansion of the critical load parameter as

$$\lambda_{cr}=1+0\varepsilon-\frac{n_{cr}^2-1}{12}\varepsilon^2+0\varepsilon^3+\ldots \quad (80)$$

And the corresponding five term expansion of the critical load may be obtained as,

$$(p_o)_{cr}=0+0\varepsilon+0\varepsilon^2+\frac{E(n_{cr}^2 1)}{12(1-\nu^2)}\varepsilon^3-\frac{E(n_{cr}^2-1)}{8(1-\nu^2)}$$

$$\varepsilon^4+\ldots \tag{81}$$

We can note that the critical load is an increasing function of the angular wavenumber n. Therefore, noting $n_{cr}\neq 0$, $n_{cr}\neq 1$, we can deduce $n_{cr}=2$.

5. Concluding Remarks

We compare our asymptotic result (81) with the classical Timosenko's result (Timosenko and Gere(1960)). Timosenko obtained the following critical load from his nonlinear theory, i.e.,

$$(p_o)_{cr}=\frac{Eh^3(n_{cr}^2-1)}{12r_i^3(1-\nu^2)},$$

which is exactly the fourth term in our result (81) with noting the definition $\varepsilon\equiv h/r_i$. Next, we compare our current result with the incompressible one computed by Triantafyllidis and Kwon(1987). For this, we should notice the following fact, i.e.,

$$\varepsilon\equiv\frac{h}{r_i}=\varepsilon^*+\frac{1}{2}(\varepsilon^*)^2+\frac{1}{6}(\varepsilon^*)^3+\ldots,$$

with $\varepsilon^*\equiv\ln\frac{r_o}{r_i}$,

and so we have

$$\lambda_{cr}=\overset{0}{\lambda}_{cr}+\overset{1}{\lambda}_{cr}\varepsilon+\overset{2}{\lambda}_{cr}\varepsilon^2+\overset{3}{\lambda}_{cr}\varepsilon^3+\ldots$$
$$=\overset{0}{\lambda}{}^*_{cr}+\overset{1}{\lambda}{}^*_{cr}\varepsilon^*+\overset{2}{\lambda}{}^*_{cr}(\varepsilon^*)^2+\overset{3}{\lambda}{}^*_{cr}(\varepsilon^*)^3+\ldots$$

with

$$\overset{0}{\lambda}{}^*_{cr}=\overset{0}{\lambda}_{cr},\ \overset{1}{\lambda}{}^*_{cr}=\overset{1}{\lambda}_{cr},\ \overset{2}{\lambda}{}^*_{cr}=\frac{1}{2}\overset{1}{\lambda}_{cr}+\overset{2}{\lambda}_{cr},$$

$$\overset{3}{\lambda}{}^*_{cr}=\frac{1}{6}\overset{1}{\lambda}_{cr}+\overset{2}{\lambda}_{cr}+\overset{3}{\lambda}_{cr}\ \text{etc.}$$

Now that the result in (81) is valid for all ν up to the ε^3 term, we have, for the incompressible solid $(\nu=1/2)$

$$\overset{0}{\lambda}_{cr}=1,\ \overset{1}{\lambda}_{cr}=0,\ \overset{2}{\lambda}_{cr}=-\frac{n_{cr}^2-1}{12}\ \text{and}\ \overset{3}{\lambda}_{cr}=0. \tag{82}$$

And so we can obtain the corresponding terms with respect to ε^* as,

$$\overset{0}{\lambda}{}^*_{cr}=\overset{0}{\lambda}_{cr}=1,\ \overset{1}{\lambda}{}^*_{cr}=\overset{1}{\lambda}_{cr}=0,$$
$$\overset{2}{\lambda}{}^*_{cr}=\overset{2}{\lambda}_{cr}=-\frac{n_{cr}^2-1}{12},$$
$$\overset{3}{\lambda}{}^*_{cr}=\overset{2}{\lambda}_{cr}+\overset{3}{\lambda}_{cr}=-\frac{n_{cr}^2-1}{12}.$$

Or,

$$\lambda_{cr}=1+0\ln\frac{r_o}{r_i}-\frac{n_{cr}^2-1}{12}\left(\ln\frac{r_o}{r_i}\right)^2-\frac{n_{cr}^2-1}{12}\left(\ln\frac{r_o}{r_i}\right)^3+\ldots n_{cr}=2, \tag{83}$$

with

which is the exactly same result that Triantafyllidis and Kwon obtained in 1987 for the incompressible case.

Now, comparing our result with the classical one, we may note that we not only obtained the same result as the classical one, but we also attained the higher order terms which cannot be obtained from the classical nonlinear theory and gives us the much higher accuracy. In such a sense, our current general theory is very powerful to compute the desired engineering accuracy.

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